Robust Tensor Decomposition via Orientation Invariant Tubal Nuclear Norms

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[1]Nanjing University of Science and Technology, China
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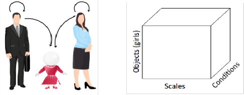
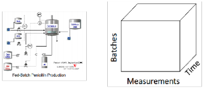
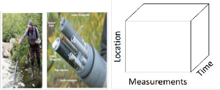
Tensor data is everywhere!



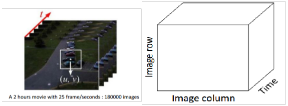
Psychology
Behavior analysis



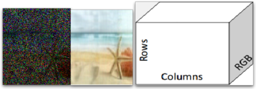
Process monitoring
Failure detection



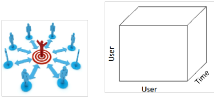
Environment monitoring
Quality assessment



Video surveillance
Anomaly detection

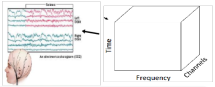


Image/Video processing
Inpainting/De-noising

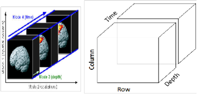


Social networks
Community detection

Question system
Topic model



EEG signal processing
Disease surveillance



MRI
Behavior recognition

Robust Tensor Decomposition (RTD)

☹ Observed tensor data are often not clean

May be corrupted by both **outliers** and **noises**

Due to: sensor failures, lens pollution, video abnormalities, corruption of images, ...

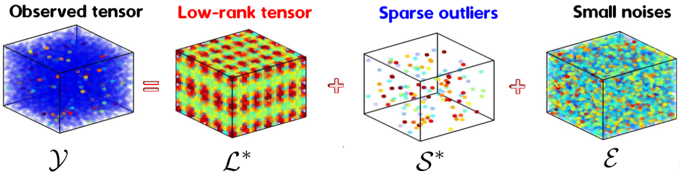
☻ Many tensor data are low-rank

E.g. images and videos have (well/approx.) **low-rank structure** (Liu J et al. PAMI 2013; Zhao QB et al. PAMI 2015)

⇓ **This paper**

An Observation Model (Gu QQ et al. NIPS 2014)

$$\mathcal{Y} = \mathcal{L}^* + \mathcal{S}^* + \mathcal{E} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$$



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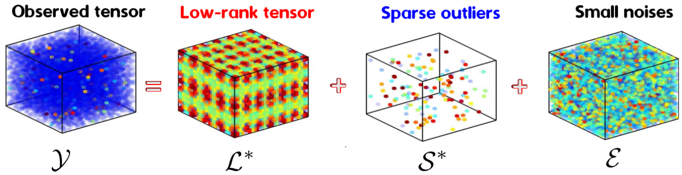

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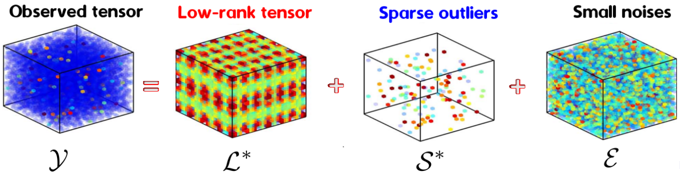

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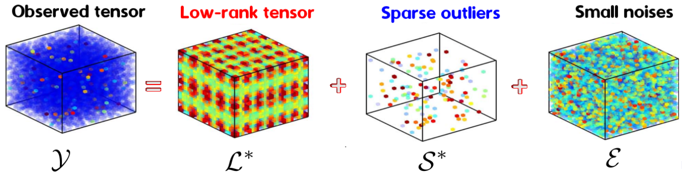

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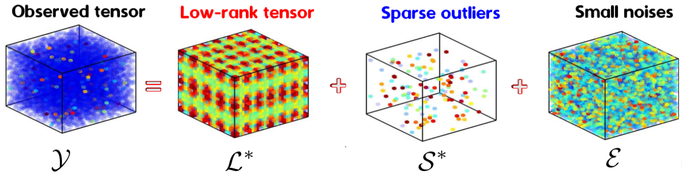

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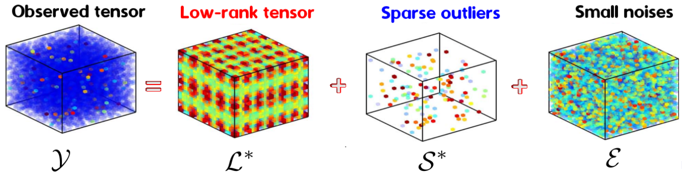

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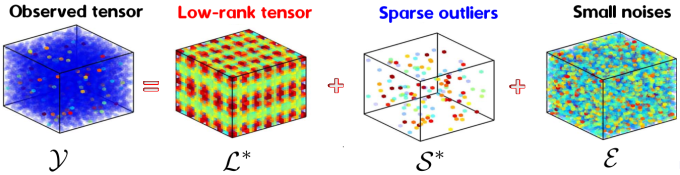

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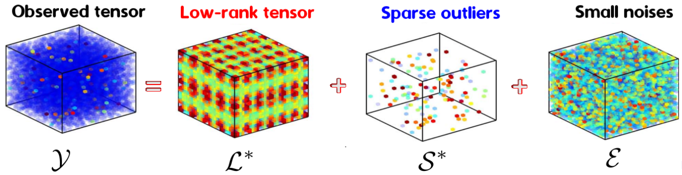

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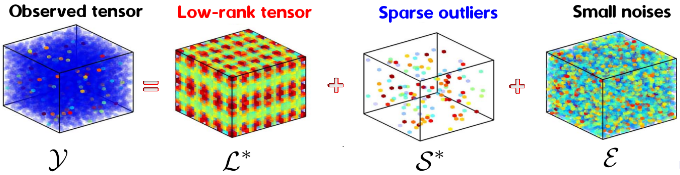
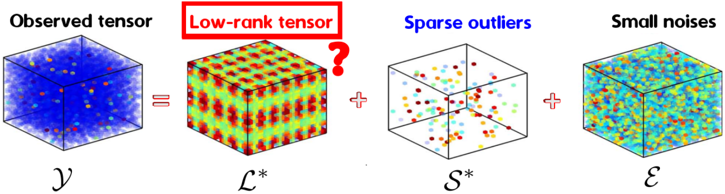


Robust Tensor Decomposition

Problem

How to estimate **the clean** $\mathcal{L}^*$ from corrupted observation $\mathcal{Y} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$?



How to exploit the low-rank structure of $\mathcal{L}^*$?

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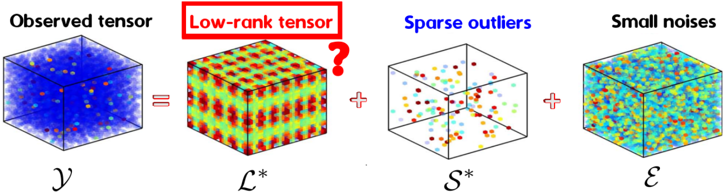
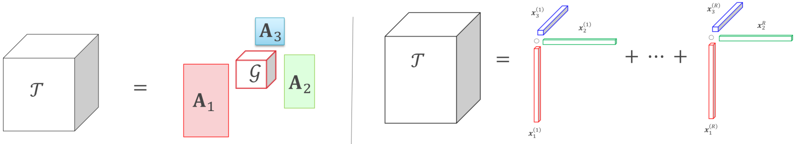


How to exploit the low-rank structure of $\mathcal{L}^*$?

Commonly used tensor low-rank strucure

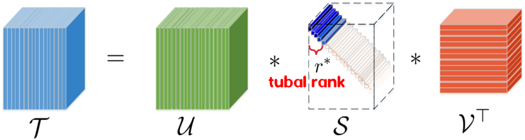


Low Tucker rank structure **Low CP rank structure**

Low-tubal-rank Structure

shown to have stronger modeling capabilities than low-Tucker-rank/low-CP-rank structure for images and videos[a]

[a] E.g. Liu XY et al. TIT 2020; Zhang ZM et al. CVPR 2014, IJCAI 2016; Lu CY et al. CVPR 2016, IJCAI 2018, PAMI 2019; Zhou P et al. CVPR 2017, PAMI 2020; Xie Y et al. IJCV 2018



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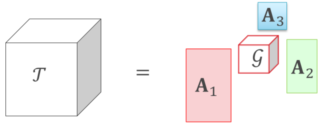

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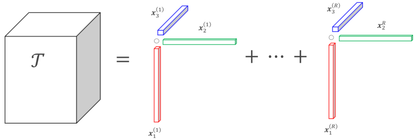

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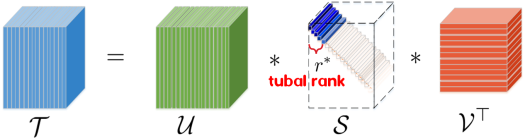

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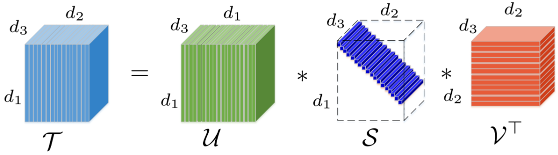

Theorem 1 (Tensor SVD (Kilmer et al. 2013)).

Any 3-way tensor $\mathcal{T} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ can be decomposed as

$$\mathcal{T} = \mathcal{U} * \mathcal{S} * \mathcal{V}^\top$$

1. $*$ is the tensor-tensor product (t-product) (Kilmer et al. 2013)
2. $\mathcal{U} \in \mathbb{R}^{d_1 \times d_1 \times d_3}$, $\mathcal{V} \in \mathbb{R}^{d_2 \times d_2 \times d_3}$ are orthogonal tensors (Kilmer et al. 2013)
3. $\mathcal{S} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ is an f-diagonal tensor (Kilmer et al. 2013)

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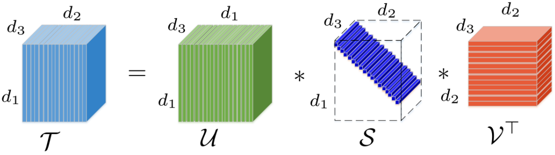

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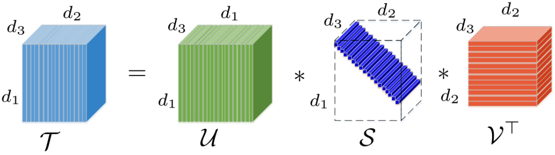

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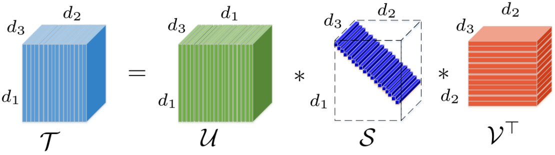

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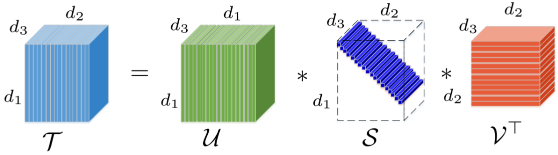

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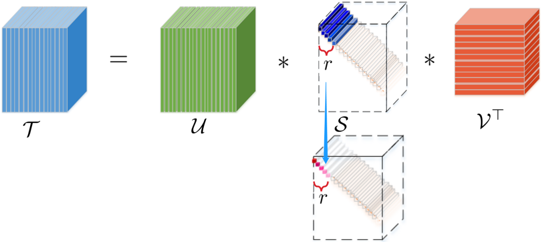
Definition 2 (Tubal Rank (Kilmer et al. 2013)).

The tubal rank of $\mathcal{T} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ is the number of non-zero tubes in $\mathcal{S}$

$$r_{\text{tb}}(\mathcal{T}) := \#\{i \mid \mathcal{S}(i,i,:) \neq \mathbf{0}\}$$



Relationship between t-product and DFT indicates (Lu CY et al. PAMI 2019):

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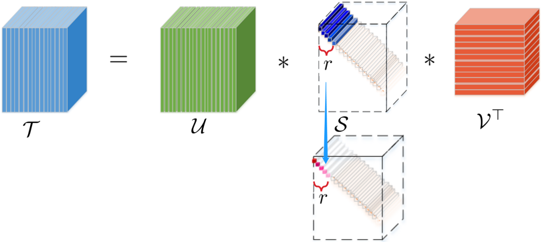

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Tensor "Singular Values"

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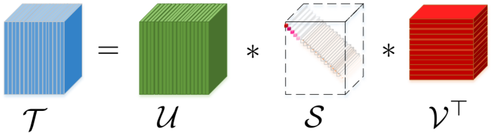
$\mathcal{S}(i,i,1)$'s are also called the "**singular values**" of tensor $\mathcal{T}$ (Lu CY et al. PAMI 2019)

Definition 3 (Tubal Nuclear Norm, TNN).

The TNN of $\mathcal{T}$ is **the sum of its singular values**

$$\|\mathcal{T}\|_{\star} := \sum_{i=1}^{d_1 \wedge d_2} \mathcal{S}(i,i,1)$$



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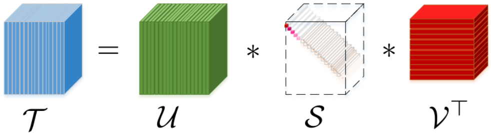

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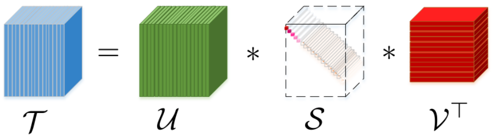
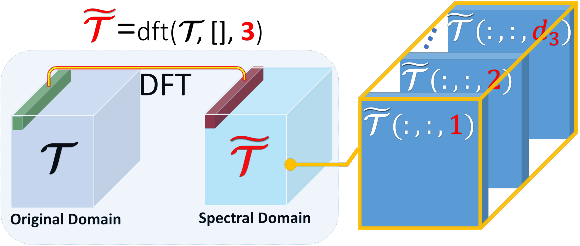


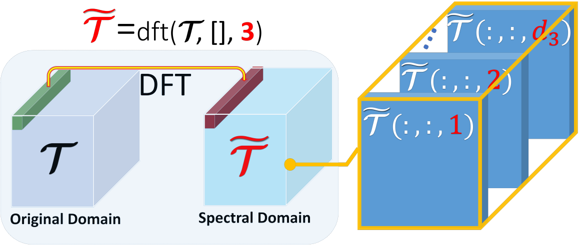


Low-rankness in spectral domain

Relationship between t-product and DFT indicates:

$$\|\mathcal{T}\|_\star = \frac{1}{d_3}\sum_{k=1}^{d_3}\|\widetilde{\mathcal{T}}(:,:,k)\|_*$$

TNN measures low-rankness in spectral domain along the 3d orientation



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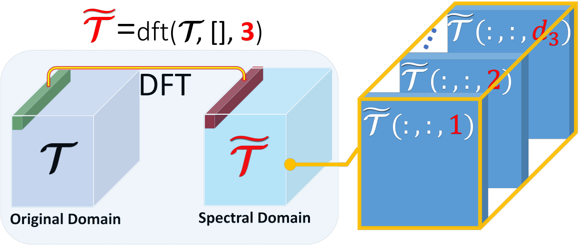

Low-rankness in spectral domain

Relationship between t-product and DFT indicates:

$$\|\mathcal{T}\|_\star = \frac{1}{d_3}\sum_{k=1}^{d_3}\|\widetilde{\mathcal{T}}(:,:,k)\|_*$$

TNN measures low-rankness in spectral domain along the 3d orientation

Weaknesses of TNN

$$\|\mathcal{T}\|_\star = \frac{1}{d_3}\sum_{k=1}^{d_3}\|\widetilde{\mathcal{T}}(:,:,k)\|_*, \quad \text{where } \widetilde{\mathcal{T}} = \mathrm{dft}(\mathcal{T},[],3) \in \mathbb{R}^{d_1\times d_2\times d_3}$$

- ☹ **Orientation sensitivity**: computed after DFT along the 3-rd orientation
- ☹ **Order limitation**: defined only for 3-way tensors

⇓ TNN fails to model

Multi-orientational spectral low-rankness for K-way $(K \geq 3)$ tensors

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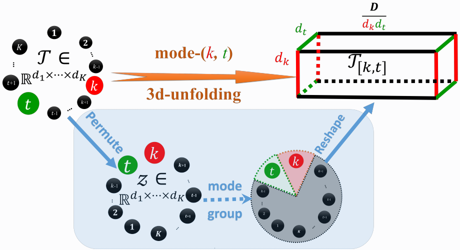
Exploiting multi-orientational spectral low-rankness

Idea: convert a K-way tensor to K 3-way tensors

then, each 3-way tensor handles one orientation

Step 1: Define mode-(k,t) 3d-unfolding



Step 2: Let $t = k+1$. Then mode t traverses all the K orientations when $k = 1 : K$.

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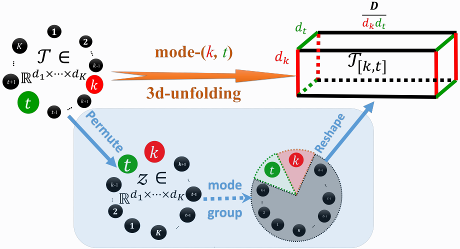

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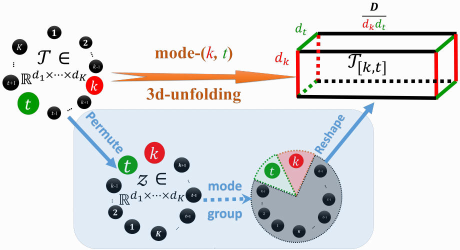

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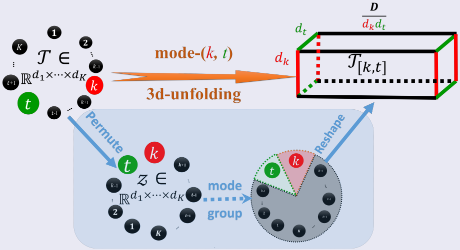

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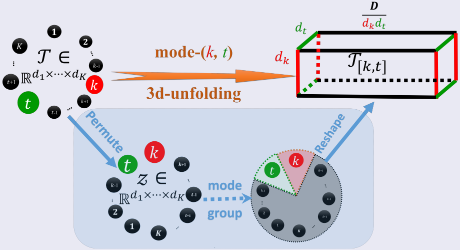

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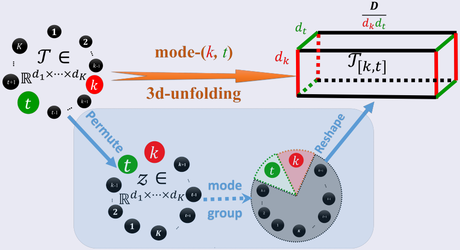

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Definition 4 (Overlapped OITNN: Sum of TNNs after unfolding).

OITNN-O of $\mathcal{T} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ is the sum of K TNNs after 3-d unfoldings

$$\|\mathcal{T}\|_{\star o} := \sum_{k=1}^{K} w_k \|\mathcal{T}_{[k]}\|_{\star},$$

with weights $\sum_k w_k = 1$.

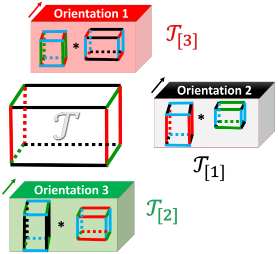


Figure 1: OITNN-O encourages simultaneous low-tubal-rankness in all orientations

Definition 5 (Latent OITNN: Sum of TNNs after decomposition).

OITNN-L of $\mathcal{T} \in \mathbb{R}^{d_1 \times \cdots \times d_K}$ is the infimum of sum of K TNNs among all decompositions

$$\|\mathcal{T}\|_{\star\iota} := \inf_{\sum_k \mathcal{L}^{(k)} = \mathcal{T}} \sum_{k=1}^{K} v_k \|\mathcal{L}^{(k)}_{[k]}\|_{\star},$$

with weights $\sum_k v_k = 1$.

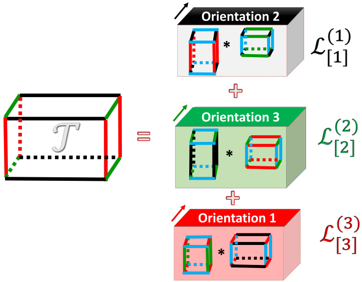
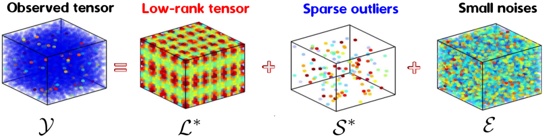


Figure 2: OITNN-L models $\mathcal{T}$ as sum of K low-tubal-rank tensors $\{\mathcal{L}^{(k)}\}$

Proposed Models for RTD



Model I: RTD based on OITNN-O

$$(\hat{\mathcal{L}}_o, \hat{\mathcal{S}}_o) \in \operatorname*{argmin}_{\mathcal{L},\mathcal{S}} \frac{1}{2}\|\mathcal{Y} - \mathcal{L} - \mathcal{S}\|_F^2 + \lambda_o \|\mathcal{L}\|_{\star o} + \mu_o \|\mathcal{S}\|_1$$

$$\text{s.t.} \quad \|\mathcal{L}\|_\infty \le \alpha \quad \leftarrow \text{(incoherence condition)}$$

Model II: RTD based on OITNN-L

$$(\{\hat{\mathcal{L}^{(k)}}\}, \hat{\mathcal{S}}_\iota) \in \operatorname*{argmin}_{\{\mathcal{L}^{(k)}\},\mathcal{S}} \frac{1}{2}\|\mathcal{Y} - \mathcal{L} - \mathcal{S}\|_F^2 + \lambda_\iota \sum_k v_k \|\mathcal{L}^{(k)}_{[k]}\|_\star + \mu_\iota \|\mathcal{S}\|_1$$

$$\text{s.t.} \quad \|\mathcal{L}^{(l)}_{[k]}\| \le \beta \tilde{d}_k, \ \forall l \ne k; \|\sum_k \mathcal{L}^{(k)}\|_\infty \le \alpha \leftarrow \text{(incoherence condition)}$$

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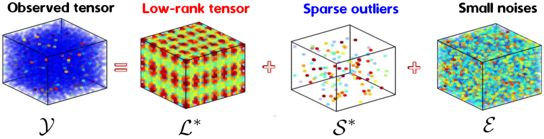

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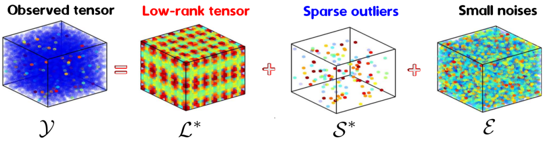

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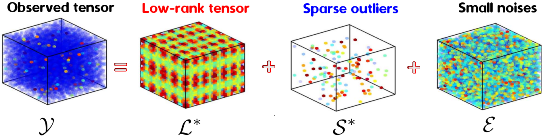

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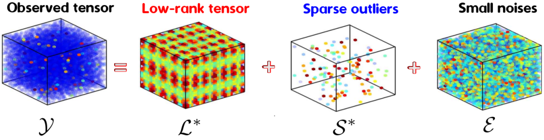

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Bounds on the Estimation Error

When noise tensor $\mathcal{E}$ has i.i.d. $\mathcal{N}(0, \sigma^2)$ entries

For $\mathcal{L}^* \in \mathbb{R}^{d \times d \times \cdots \times d}$, it holds w.h.p. after parameter tuning:

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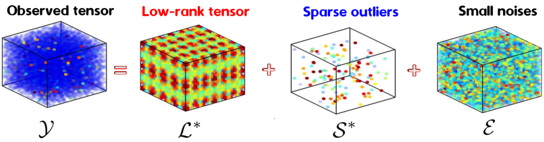
$\Downarrow$

- ✓ Bound on Model I: controlled by spectral low-rankness of **all orientations**
- ✓ Bound on Model II: controlled by **the orientation with lowest rank** in spectral domain



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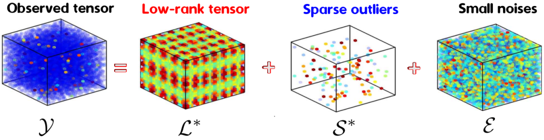

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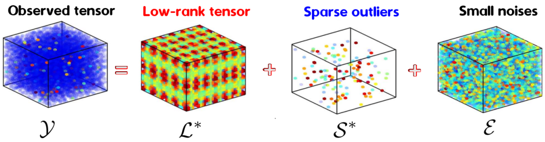

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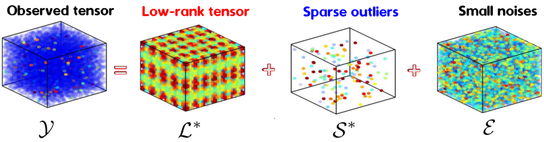

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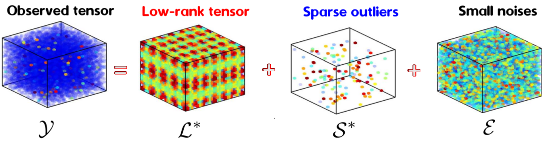
$$\frac{\|\sum_k \hat{\mathcal{L}}^{(k)} - \mathcal{L}^*\|_{\mathsf{F}}^2 + \|\hat{\mathcal{S}}_{\iota} - \mathcal{S}^*\|_{\mathsf{F}}^2}{d^K} \lesssim \sigma^2\Big(d^{-1}\min_k\{r_{\mathsf{tb}}(\mathcal{L}^*_{[k]})\} + \|\mathcal{S}^*\|_{l_0} K\log d\Big) \quad \leftarrow \text{(Model II)}$$

$\Downarrow$

- ✓ Bound on Model I: controlled by spectral low-rankness of all orientations
- ✓ Bound on Model II: controlled by the orientation with lowest rank in spectral domain



Bounds on the Estimation Error

When noise tensor $\mathcal{E}$ has i.i.d. $\mathcal{N}(0,\sigma^2)$ entries

For $\mathcal{L}^* \in \mathbb{R}^{d\times d\times\cdots\times d}$, it holds w.h.p. after parameter tuning:

$$\frac{\|\hat{\mathcal{L}}_{\mathsf{o}} - \mathcal{L}^*\|_{\mathsf{F}}^2 + \|\hat{\mathcal{S}}_{\mathsf{o}} - \mathcal{S}^*\|_{\mathsf{F}}^2}{d^K} \lesssim \sigma^2\Big(d^{-1}K^{-2}\sum_k r_{\mathsf{tb}}(\mathcal{L}^*_{[k]}) + \|\mathcal{S}^*\|_{l_0} K\log d\Big) \quad \leftarrow \text{(Model I)}$$

$$\frac{\|\sum_k \hat{\mathcal{L}}^{(k)} - \mathcal{L}^*\|_{\mathsf{F}}^2 + \|\hat{\mathcal{S}}_{\iota} - \mathcal{S}^*\|_{\mathsf{F}}^2}{d^K} \lesssim \sigma^2\Big(d^{-1}\min_k\{r_{\mathsf{tb}}(\mathcal{L}^*_{[k]})\} + \|\mathcal{S}^*\|_{l_0} K\log d\Big) \quad \leftarrow \text{(Model II)}$$

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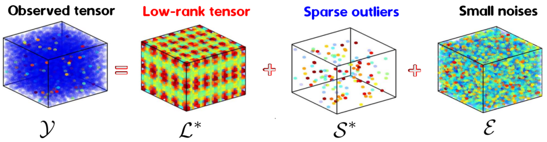
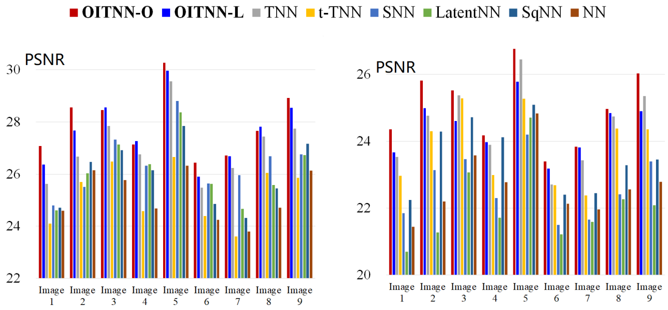


Robust Image Recovery



(a) $(\mathfrak{s}, c) = (0.05, 0.1)$ (b) $(\mathfrak{s}, c) = (0.15, 0.15)$

Figure 3: Robust image recovery with different corruption ratio $\mathfrak{s}$ and noise level c.

Image Completion

1. Setting I: 90% random missing
2. Setting II: rows and columns missing, total ratio 85%

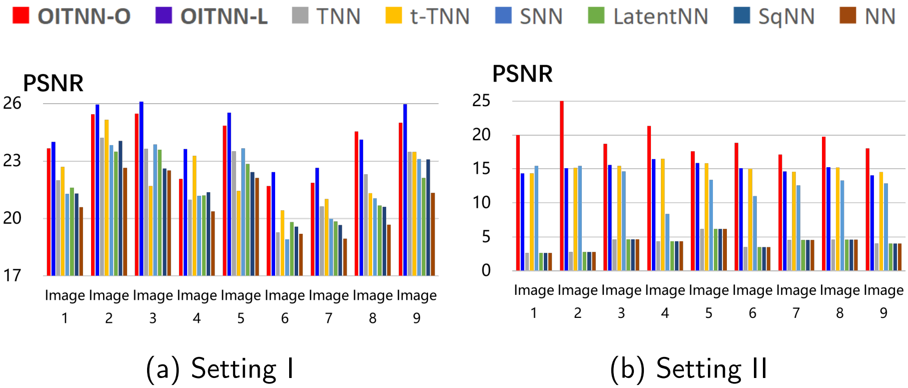
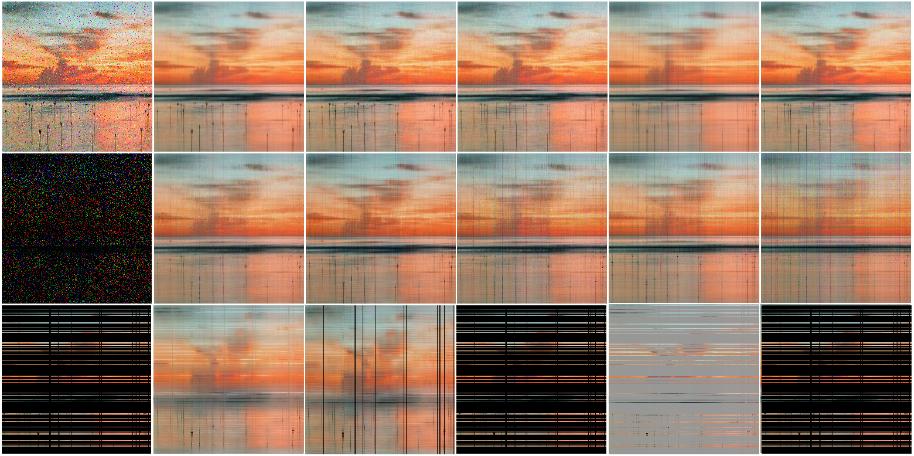
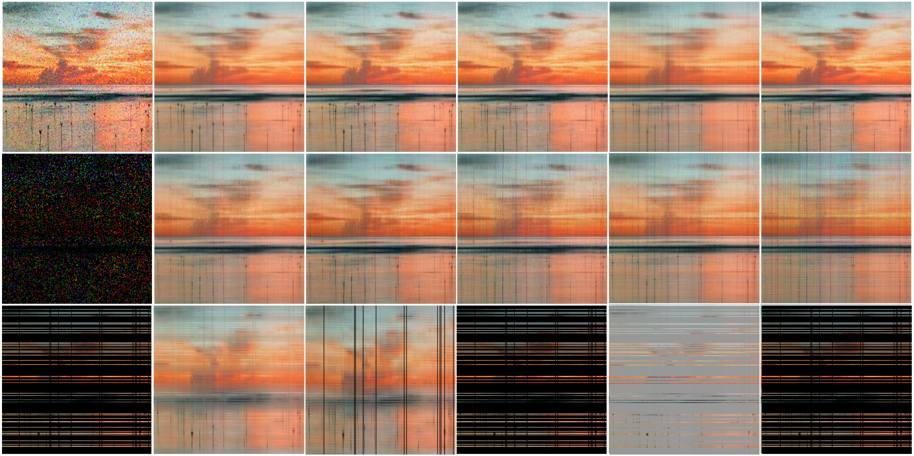


Figure 4: Quantitative comparison in image completion.



(a) Observation (b) OITNN-O (c) OITNN-L (d) TNN (e) SNN (e) LatentNN

1. Row 1: robust image recovery with corruption ratio $\mathfrak{s} = 0.05$ and noise level $c = 0.1$
2. Row 2: image completion with 90% random missing entries
3. Row 3: image completion with missing columns and rows (total missing ratio 85%)



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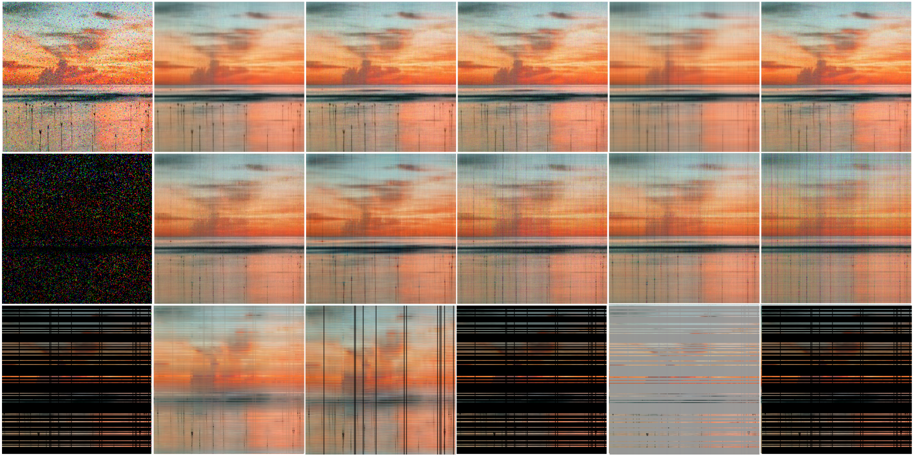

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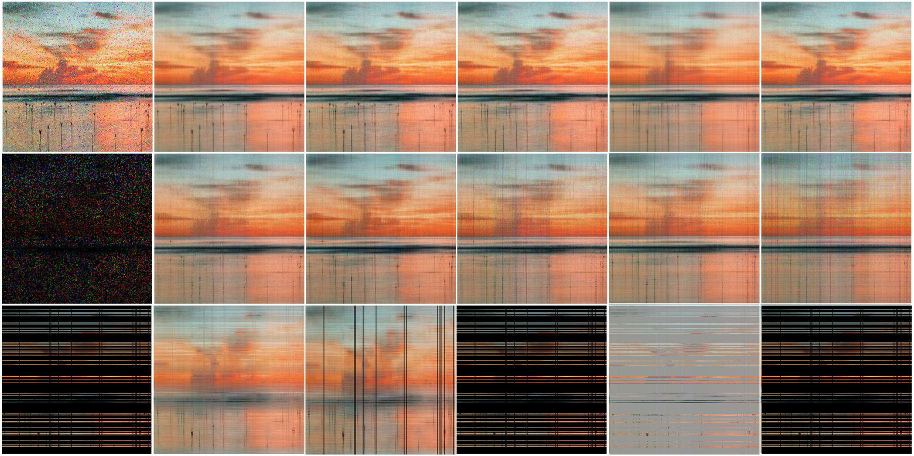

(a) Observation (b) OITNN-O (c) OITNN-L (d) TNN (e) SNN (e) LatentNN

1. Row 1: robust image recovery with corruption ratio $\mathfrak{s} = 0.05$ and noise level $c = 0.1$
2. Row 2: image completion with 90% random missing entries
3. Row 3: image completion with missing columns and rows (total missing ratio 85%)

Video Completion

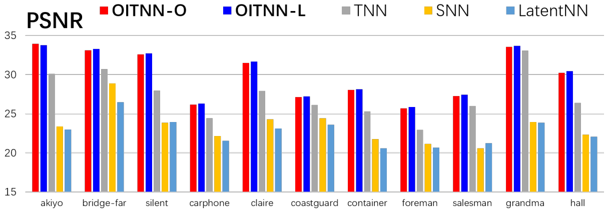

Figure 5: Video completion with 90% random missing

Conclusion

Contributions

1. We defined **two new norms** for K-way $(K \geq 3)$ tensors.
2. We presented **two models for RTD** with error bounds.

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